Real Analysis Qualifier Exam–August 2004

1. (**Do ONLY four from the following five statements**) Prove or disprove, by a counterexample, the following statements:

(a) (*5 points*) A countable subset of $\mathbb{R}$ has Lebesgue measure zero.

(b) (*5 points*) If a subset of $\mathbb{R}$ has Lebesgue measure zero then it is countable.

(c) (*5 points*) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is integrable then so is f^2.

(d) (*5 points*) If $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ are Lebesgue measurable functions such that $f_n \to f$ a.e. then $f_n \to f$ in measure.

(e) (*5 points*) There is a Lebesgue measurable set which is not in the Borel σ-algebra.

Do ONLY one of the Problems 2 and 3.

2. (*10 points*) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is continuous then it maps closed sets into Lebesgue measurable sets.

3. (*10 points*) If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is such that $|f(x) - f(y)| \leq 8|x - y|$, for all $x, y \in \mathbb{R}$, then it maps sets of Lebesgue measure zero into sets of Lebesgue measure zero.

Do ONLY one of the Problems 4 and 5.

4. (*10 points*) Let $(X, \mathcal{M}, \mu)$ be a measure space, and $f : X \longrightarrow (0, \infty)$ be a measurable function. Show that $\dfrac{1}{f}$ is also measurable.

5. (*10 points*) Let $(X, \mathcal{M}, \mu)$ be a measure space. If $f_n : X \to \mathbb{R}$ is a sequence of measurable functions on X then the set of all points $x \in X$ where $\lim_{n\to\infty} f_n(x)$ exists is a measurable set.

6. (*15 points*) (Dominated Convergence Theorem) Let $(X, \mathcal{M}, \mu)$ be a measure space and $\{f_n\}$, and g be integrable functions such that:

(a) $f_n \longrightarrow f$ a.e. in X,

(b) $|f_n| \leq g$, for all n.

Then show that

$$\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n.$$

Do ONLY one of the Problems 7 and 8.

7. (*15 points*) Compute the following limit. Justify your steps.

$$\lim_{n\to\infty}\int_0^\infty \frac{8n^4+\cos(n^2x^2+1)}{4n^4+x^4}\, e^{-3x}dx$$

8. (*15 points*) If $f:\mathbb{R}\longrightarrow\mathbb{R}$ is a Lebesgue integrable function then show that its Fourier transform

$$\widehat{f}(\xi)\doteq\int_{\mathbb{R}} e^{-ix\xi}f(x)dx,\ \ \xi\in\mathbb{R},$$

is a continuous function.

Do ONLY one of the Problems 9 and 10.

9. (*15 points*) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a Lebesgue integrable function. Show that for any $\varepsilon > 0$ there is a $\delta > 0$, depending on ε, such that

$$\int_A |f(x)| dx < \varepsilon,$$

for any Lebesgue measurable set A with $m(A) < \delta$.

10. (*15 points*) Let $(X, \mathcal{M}, \mu)$ be a measure space. Prove that for any $\epsilon > 0$ and any integrable function $f : X \to \mathbb{R}$, one has

$$\mu\{x : |f(x)| \geq \epsilon\} \leq \frac{1}{\epsilon} \int_X |f| \, d\mu.$$

Do ONLY one of the Problems 11 and 12.

11. (*15 points*) State and prove Fatou's Lemma.

12. (*15 points*) Give a clear and brief description of the construction of the Lebesgue measure in $\mathbb{R}$, starting with the elementary class of the half-open intervals. Do not give proofs.